# 11-3 Decomposing Into Partial Fractions

Learning Objectives:

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# I. Proper Fractions with Distinct Linear Factors

$$\frac{5-x}{(2x-1)(x+1)} = \frac{A}{(2x-1)} + \frac{B}{(x+1)}$$

**Ex1.** $\displaystyle\int \frac{5-x}{2x^2+x-1}dx$

$$\int \frac{5-x}{(2x-1)(x+1)} \rightarrow 5-x=(x+1)A+(2x-1)B$$

when $x=-1$, $A=3$

when $x=\frac{1}{2}$, $B=-2$

$$\rightarrow \int \frac{3}{2x-1} - \frac{2}{x+1} \rightarrow \boxed{\frac{3}{2}\ln|2x-1|-2\ln|x+1|+c}$$

$$\ln|2x-1|^{3/2} - \ln|x+1|^2$$

$$\ln\left|\frac{(2x-1)^{3/2}}{(x+1)^2}\right| + C$$

# II. Improper Fractions with Distinct Linear Factors

Ex2. $\int \frac{x^3 - x + 5}{x^2 + x - 2} dx$

$$x - 1 + \frac{2x+3}{x^2+x-2}$$

$$x^2 + x - 2 \overline{\big)\, x^3 + 0x^2 - x + 5}$$

$$-x^3 - x^2 + 2x$$

$$-x^2 + 1x + 5$$

$$+x^2 + 1x - 2$$

$$2x + 3$$

$$\int x - 1 + \boxed{\frac{2x+3}{x^2+x-2}} \, dx \leftarrow \text{decompose}$$

$$\frac{2x+3}{(x+2)(x-1)}$$

$$\frac{A}{(x+2)} + \frac{B}{(x-1)} = \frac{2x+3}{(x+2)(x-1)}$$

$$A(x-1) + B(x+2) = 2x+3$$

$$A = \frac{1}{3} \quad B = \frac{5}{3}$$

$$\int x - 1\, dx + \frac{1}{3}\int \frac{dx}{x+2} + \frac{5}{3}\int \frac{dx}{x-1}$$

$$\boxed{\frac{1}{2}x^2 - x + \frac{1}{3}\ln|x+2| + \frac{5}{3}\ln|x-1| + c}$$

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# III. Repeated Linear Factors

## Ex3. Integrate:

1.) $\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$

$x(x^2+2x+1)$
$x\ (x+1)(x+1)$

$$\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$5x^2+20x+6 = A(x+1)^2 + B(x)(x+1) + C(x)$$

if $x=-1$ $\quad -9 = A(0) + B(0) + C(-1)$
$-9 = -C \Rightarrow \underline{C = 9}$

if $x=0$ $\quad 6 = A(1)^2 + B(0) + C(0)$
$\underline{6 = A}$

$$5x^2+20x+6 = 6(x+1)^2 + B(x)(x+1) + 9x$$

if $x=1$ $\quad 31 = 24 + 2B + 9$
$-2 = 2B$
$-1 = B$

$$\int \frac{6}{x} + \frac{-1}{x+1} + \frac{9}{(x+1)^2}dx$$

$9(x+1)^{-2}$
$-9(x+1)^{-1}$

$$6\ln|x| - \ln|x+1| - \frac{9}{(x+1)} + C$$

2.) $\int \frac{x^2+3x-4}{x^3+-4x^2+4x}dx$

$x^2-4x+4$

ex $\int \frac{x^2+3x-4}{x^3-4x^2+4x}dx \qquad \frac{x^2+3x-4}{x(x-2)^2} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$

$x^2+3x-4 = A(x-2)^2 + B(x(x-2)) + Cx$

$x=0 \qquad -4 = 4A \qquad A=-1$

$x=2 \qquad \cancel{4+6-4} \quad 6=2C \qquad C=3$

$1+3-4 = -1(-1)^2 + B(1(-1)) + 3$

$0 = -1 - B + 3$

$-2 = -B \qquad B=2$

$\frac{-1}{x}+\frac{2}{x-2}+\frac{3}{(x-2)^2} \qquad 3(x-2)^{-2} \qquad -3(x-2)^{-1}$

$\boxed{-\ln|x| + 2\ln|x-2| + -3(x-2)^{-1} + C}$

IV. Distinct Linear and Quadratic Factors

Ex4. $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx = \int \frac{2x^3-4x-8}{x(x-1)(x^2+4)}dx$

$\frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$

$2x^3-4x-8 = A(x-1)(x^2+4) + B\cdot x(x^2+4) + (Cx+D)x(x-1)$

$x=1$ $\quad -10 = A\cdot(0) + B(1)(5) + (Cx+D)\cdot 0$

$-10 = 5B \qquad B=-2$

$x=0$ $\quad -8 = A(-1)(4) + B(0) + (Cx+D)\cdot 0$

$-8=-4A \qquad A=2$

$x=2$ $\quad 0 = 2(1)(8) + -2(2)(8) + (2C+D)\cdot 2\cdot 1$

$0 = 16 - 32 + (2C+D)\cdot 2$

$0 = -16 + 4C + 2D$

$16 = 4C + 2D$

$8 = 2C + D$

$x=-1$ $\quad -6 = 2(-2)(5) - 2(-1)(5) + (-C+D)(-1)(-2)$

$-6 = -20 + 10 + 2(-C+D)$

$-6 = -10 - 2C + 2D$

$4 = -2C + 2D$

Linear Combo

$8 = \cancel{2C} + D$

$+\ 4 = \cancel{-2C} + 2D$

$12 = 3D$

$D=4$

$4 = -2C + 2(4)$

$4 = -2C + 8$

$-4 = -2C$

$C=2$

Substitution

$8 = 2C + D \Rightarrow D = 8-2C$

$4 = -2C + 2D$

$4 = -2C + 2(8-2C)$

$4 = -2C + 16 - 4C$

$4 = 16 - 6C$

$-12 = -6C$

$C=2$

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$$= \int \frac{2}{x} + \frac{-2}{x-1} + \frac{2x+4}{x^2+4}\,dx$$

$$= \int \frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\,dx$$

$$= \int \frac{2}{x} - \frac{2}{x-1}\,dx + \int \frac{2x}{x^2+4}\,dx + 4\int \frac{1}{x^2+4}\,dx$$

$$= 2\ln|x| - 2\ln|x-1|$$

$u = x^2 + 4$

$\frac{du}{dx} = 2x \quad dx = \frac{du}{2x}$

$\int \frac{\cancel{2x}}{u} \cdot \frac{du}{\cancel{2x}}$

$\int \frac{1}{u}\,du$

$\ln|u|$

$\ln|x^2+4|$

$u = \frac{x}{2}$

$\sqrt{x^2} = \sqrt{4u^2}$

$x = 2u$

$dx = 2du$

$4\int \frac{2du}{4u^2+4}$

$8\int \frac{du}{4(u^2+1)}$

$2\int \frac{du}{u^2+1}$

$2\tan^{-1} u$

$2\tan^{-1}\left(\frac{x}{2}\right)$

$$2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

## Repeated Linear and Quadratic Factors

Ex4. $\int \frac{8x^3+13x}{(x^2+2)^2}dx$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

$$8x^3+13x = (Ax+B)(x^2+2) + Cx+D$$

$$8x^3+13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$

$$8x^3 + 0x^2 + 13x = Ax^3 + Bx^2 + (2A+C)x + (2B+D)$$

$A=8$ $\quad B=0$

$2A+C=13$
$2(8)+C=13$
$16+C=13$
$C=-3$

$2B+D=0$
$2(0)+D=0$
$D=0$

$$\int \frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2}\,dx$$

$u=x^2+2$

$\frac{du}{dx}=2x$

$\frac{du}{2x}=dx$

$\int \frac{8x}{u}\cdot\frac{du}{2x}$

$\int \frac{4}{u}\,du$

$4\int \frac{1}{u}\,du$

$4\ln|u|$

$4\ln|x^2+2|$

$u=x^2+2$
$\frac{du}{dx}=2x$
$dx=\frac{du}{2x}$
$\frac{-3x}{u^2}\,\frac{du}{2x}$
$\frac{-3}{2}\int u^{-2}du$
$\frac{3}{2}\,\frac{1}{u}+C$
$\frac{3}{2}\left(\frac{1}{x^2+2}\right)+C$

$$4\ln|x^2+2| + \frac{3}{2(x^2+2)} + C$$

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# Homework

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